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CERTAIN IDENTITIES DEDUCED FROM SEARS TRANSFORMATION

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Abstract: In this paper certain identities have been deduced from the Sears Transformation formula.

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1. Introduction, Notations and Definitions

As usual for complex numbers a and q, $|q| < 1$, the $q-$shifted factorial is defined as,

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \quad n > 0,$$

we also write it as,

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \Rightarrow (a;q)_0 = 1,$$

where

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r).$$

Also, for brevity we write,

$$(a_1,a_2,...,a_r;q)_n = (a_1;q)_n(a_2;q)_n...(a_r;q)_n$$

and

$$(a_1,a_2,...,a_r;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty...(a_r;q)_\infty.$$

A basic hypergeometric series is defined as,

$${}_{r+1}\Phi_r\left[\begin{array}{l} a_1,a_2,...,a_{r+1};q;z \\ b_1,b_2,...,b_r \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,...,a_{r+1};q)_n}{(q,b_1,b_2,...,b_r;q)_n}z^n, \quad |z|<1.$$

2. Identities

In this section we deduce certain identities from Sears transformation. Some of the results of Ramanujan in his ‘Lost’ Notebook are deducible from Sears transformation. For detail about Ramanujan’s results deducible from Sears transformation one is referred [2; chapter 2] and Singh et al. [5, 6, 7].

Sears transformation is,

$${}_3\Phi_2\left[\begin{array}{l} a,b,c;q;\frac{de}{abc} \\ d,e \end{array}\right] = \frac{(e/a,de/bc;q)_\infty}{(e,de/abc;q)_\infty}\,{}_3\Phi_2\left[\begin{array}{l} a,d/b,d/c;q;\frac{e}{a} \\ d,de/bc \end{array}\right] \tag{2.1}$$

where $\left|\frac{de}{abc}\right| < 1$ and $|e/a| < 1$.

[3; App. III (III.9)]

(i) Taking $c=e$, $a,b\to\infty$ and $d=q$ in (2.1) we get

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}(q/c;q)_n c^n}{(q;q)_n^2} \tag{2.2}$$

For $c=q$, (2.2) yields,

$$\sum_{n=0}\frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty}, \tag{2.3}$$

which is a well known identity related to the partition theory [4; (6.4.1) p. 226]. Putting $c = -q$ in (2.2) we get

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{2}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_{n-1}}{(q;q)_n^2}. \tag{2.4}$$

(ii) Taking $a, b \to \infty$, $e = c = -q = d$ in (2.1) we get

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-1)^n}{(q^2;q^2)_n} = \frac{1}{(-q;q)_\infty} = (q;q^2)_\infty. \tag{2.5}$$

Comparing (2.3) and (2.4) we find

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1;q)_n}{(q;q)_n} = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{1}{(q;q)_\infty (q;q^2)_\infty}. \tag{2.6}$$

(iii) Taking $a, b \to \infty$, $d = e = -q$ and $c = q$ in (2.1) we get

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = \frac{2}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n(1+q^n)}, \tag{2.7}$$

$f(q)$ is a mock-theta function of order three [1; (2.1) page 52].
(iv) Putting $c = -1$ in (2.2) we get

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{2(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(-q;q)_n}{(q;q)_n^2}. \tag{2.8}$$

(v) Taking $a, b, c \to \infty$ in (2.1) we find

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n-1)/2}(de)^n}{(q,d,e;q)_n} = \frac{1}{(e;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} e^n}{(q,d;q)_n} \tag{2.9}$$

Taking $d = e = q$ in (2.9) we find,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{(q;q)_n^3} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n^2}. \tag{2.10}$$

Putting $d = q$, $e = -q$ in (2.9) we have

$$\sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}}{(q;q)_n^2(-q;q)_n} = \frac{1}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n^2}. \tag{2.11}$$

Again, taking $d=-q$ and $e=q$ in (2.9) we find

$$\sum_{n=0}^{\infty}\frac{q^{n(3n+1)/2}}{(q;q)_n^2(-q;q)_n}=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{(q^2;q^2)_n}. \tag{2.12}$$

Comparing (2.11) and (2.12) we find,

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q)_n^2}=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{(q^2;q^2)_n}. \tag{2.13}$$

Putting $d=e=-q$ in (2.9) we have

$$\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}}{(q^2;q^2)_n(-q;q)_n}=\frac{1}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^2;q^2)_n}. \tag{2.14}$$

Taking $d=e=-1$ in (2.9) we get

$$\sum_{n=0}^{\infty}\frac{(-1)^nq^{3n(n-1)/2}}{(q;q)_n(-1;q)_n^2}=\frac{1}{2(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n(-1;q)_n}. \tag{2.15}$$

Putting $d=\omega q$ and $e=\omega^2q$ in (2.9) we get,

$$\sum_{n=0}^{\infty}\frac{(-1)^nq^{3n(n+1)/2}}{(q;q)_n(1+q+q^2)...(1+q^n+q^{2n})}=\frac{1}{(\omega^2q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}\omega^{2n}}{(q,\omega q;q)_n}, \tag{2.16}$$

where ω is the cube root of unity.
Again putting $d=-\omega q$ and $e=-\omega^2q$ in (2.9) we get

$$\sum_{n=0}^{\infty}\frac{(-1)^nq^{3n(n+1)/2}}{(q;q)_n(1-q+q^2)...(1-q^n+q^{2n})}=\frac{1}{(-\omega^2q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}\omega^{2n}}{(q,-\omega q;q)_n}, \tag{2.17}$$

where $\omega=e^{2\pi i/3}$.
For $a=q$, $a,b\to\infty$, $d=e=-q$, (1.1) yields

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_\infty^2}=\frac{1}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1;q)_nq^{n(n+1)/2}}{(-q^2;q^2)_n}. \tag{2.18}$$

replacing q by q^2 in (2.1) we get

$$\sum_{n=0}^{\infty}\frac{(a,b,c;q^2)_n(de/abc)^n}{(q^2,d,e;q^2)_n}=\frac{(e/a,de/bc;q^2)_\infty}{(e,de/abc;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(a,d/b,d/c;q^2)_n}{(q^2,d,de/bc;q^2)_n}\left(\frac{e}{a}\right)^n. \tag{2.19}$$

Taking $c = q^2$, $b = e$, $a \to \infty$ in (2.19) we find,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)}}{(d;q^2)_n} \left(\frac{d}{q^2}\right)^n = \frac{(d/q^2;q^2)_\infty}{(e;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)} (d/e, d/q^2;q^2)_n}{(q^2, d, d/q^2;q^2)_n} e^n. \quad (2.20)$$

Putting $d = -q^2$ in (2.20) we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n(n-1)}}{(-q^2;q^2)_n} &= \frac{(-1;q^2)_\infty}{(e;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)} (-q^2/e, -1;q^2)_n}{(q^2, -q^2, -1;q^2)_n} e^n \\ &= \frac{(-1;q^2)_\infty}{(e;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)} (-q^2/e;q^2)_n e^n}{(q^4;q^4)_n}. \end{aligned} \quad (2.21)$$

For $e = -q$, (2.21) yields,

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}}{(-q^2;q^2)_n} = \frac{(-1;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2} (q;q^2)_n}{(q^4;q^4)_n}. \quad (2.22)$$

Putting $e = 0$ in (2.21) we find,

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}}{(-q^2;q^2)_n} = (-1;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2}}{(q^4;q^4)_n} = 2(-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2}}{(q^4;q^4)_n}. \quad (2.23)$$

Certain other identities can also be deduced.

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